Ejercicio 1º

Dado un campo escalar ϕ definido en 3 puntos ($\phi 1, \phi 2, \phi 3$) y una magnitud definida como: COSA=$-6\phi_1^2 - 6\phi_2^2 - 6\phi_3^2 - \sqrt{2}\phi_1\phi_2 - \sqrt{2}\phi_2\phi_3$

a) Calcular matriz A

$$COSA = (\phi_1 \quad \phi_2 \quad \phi_3) \cdot \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \cdot \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = -6\phi_1^2 - 6\phi_2^2 - 6\phi_3^2 - \sqrt{2}\phi_1\phi_2 - \sqrt{2}\phi_2\phi_3$$

Al ser A una matriz simétrica, es fácil ver que los términos cuadráticos corresponden a la diagonal y de los cruzados tenemos dos iguales de cada tipo de cruce, siendo nulo el cruce del primero y tercero, entonces tenemos:

$$(\phi_1 \quad \phi_2 \quad \phi_3) \cdot \begin{pmatrix} -6 & -\sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & -6 & -\sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & -6 \end{pmatrix} \cdot \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = -6\phi_1^2 - 6\phi_2^2 - 6\phi_3^2 - \sqrt{2}\phi_1\phi_2 - \sqrt{2}\phi_2\phi_3$$

Comprobémoslo:

$$(\phi_1 \quad \phi_2 \quad \phi_3) \cdot \begin{pmatrix} A_{11}\phi_1 + A_{12}\phi_2 + A_{13}\phi_3 \\ A_{21}\phi_1 + A_{22}\phi_2 + A_{23}\phi_3 \\ A_{31}\phi_1 + A_{32}\phi_2 + A_{33}\phi_3 \end{pmatrix} = -6\phi_1^2 - 6\phi_2^2 - 6\phi_3^2 - \sqrt{2}\phi_1\phi_2 - \sqrt{2}\phi_2\phi_3$$

$$A_{11}\phi_1^2 + A_{12}\phi_1\phi_2 + A_{13}\phi_1\phi_3 + A_{21}\phi_2\phi_1 + A_{22}\phi_2^2 + A_{23}\phi_2\phi_3 + A_{31}\phi_3\phi_1 + A_{32}\phi_3\phi_2 + A_{33}\phi_3^2$$
$$= -6\phi_1^2 - 6\phi_2^2 - 6\phi_3^2 - \sqrt{2}\phi_1\phi_2 - \sqrt{2}\phi_2\phi_3$$

$$A_{11}\phi_1^2 + A_{22}\phi_2^2 + A_{33}\phi_3^2 + \left(A_{12} + A_{21}\right)\phi_1\phi_2 + \left(A_{13} + A_{31}\right)\phi_1\phi_3 + \left(A_{23} + A_{32}\right)\phi_2\phi_3$$
$$= -6\phi_1^2 - 6\phi_2^2 - 6\phi_3^2 - \sqrt{2}\phi_1\phi_2 - \sqrt{2}\phi_2\phi_3$$

$$A_{11} = A_{22} = A_{33} = -6; \;\; A_{12} = A_{21} = A_{23} = A_{32} = -\sqrt{2}/2; \;\; A_{13} = A_{31} = 0 \;\; (matriz\ simétrica)$$

b) Diagonalizad A

det$|A - \lambda I| = 0$

$$\det \begin{vmatrix} -6-\lambda & -\sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & -6-\lambda & -\sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & -6-\lambda \end{vmatrix} = 0 \;\Rightarrow -(6+\lambda)^3 + (6+\lambda) = 0 \;\; \Rightarrow \;\; \left((6+\lambda)^2 - 1\right)(6+\lambda) = 0$$

$$\lambda_2 = -6 \;\; y \;\; \lambda^2 + 12\lambda + 35 = 0 \Rightarrow \; \lambda_{1,3} = \frac{-12 \pm \sqrt{144 - 140}}{2} \Rightarrow \; \lambda_1 = -5 \; y \; \lambda_3 = -7$$

$\lambda_1 = -5 \;\Rightarrow\; x = z = -\dfrac{y}{\sqrt{2}}$; $\lambda_2 = -6 \;\Rightarrow\; x = -z$ e $y = 0$; $\lambda_3 = -7 \;\Rightarrow\; x = z = \dfrac{y}{\sqrt{2}}$

$$D = \begin{pmatrix} -5 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -7 \end{pmatrix}; \quad M = \begin{pmatrix} -1/2 & 1/\sqrt{2} & -1/2 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ -1/2 & -1/\sqrt{2} & -1/2 \end{pmatrix}; \ \det M = 1$$

c) Demostrar que: COSA= $-5\psi_1^2 - 6\psi_2^2 - 7\psi_3^2$

$$\text{COSA} = (\psi_1 \quad \psi_2 \quad \psi_3)\begin{pmatrix} -5 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -7 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = (\psi_1 \quad \psi_2 \quad \psi_3)\begin{pmatrix} -5\psi_1 \\ -6\psi_2 \\ -7\psi_3 \end{pmatrix} = -5\psi_1^2 - 6\psi_2^2 - 7\psi_3^2$$

Se podría obtener a partir de la siguiente relación sustituyendo en las variables iniciales de la magnitud COSA:

$$\phi_1 = -\frac{1}{2}\psi_1 + \frac{1}{\sqrt{2}}\psi_2 - \frac{1}{2}\psi_3$$
$$\phi_2 = \frac{1}{\sqrt{2}}\psi_1 - \frac{1}{\sqrt{2}}\psi_3$$
$$\phi_3 = -\frac{1}{2}\psi_1 - \frac{1}{\sqrt{2}}\psi_2 - \frac{1}{2}\psi_3$$